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THE OPTIMUM NON-CONICAL AND ASYMMETRICAL THREE-DIMENSIONAL CONFIGURATIONS†

G. Ye. YAKUNINA

Moscow

e-mail: galina_yakunina@mail.ru

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A method of constructing non-conical three-dimensional bodies of minimum drag is proposed assuming that the force of the medium acting on an element of the body surface depends only on the orientation of the element with respect to the direction of motion. For a specified base area and maximum permissible length of the body, the shape of these bodies is formed by combinations of parts of the surface of a circular cone and planes, the normal to which makes a certain optimum angle with the direction of motion. It is shown that the cross-section of the optimum body may be asymmetrical but the force acting on the body has no component in a plane perpendicular to the direction of motion. © 2000 Elsevier Science Ltd. All rights reserved.

The problem of constructing the three-dimensional shape of a body of minimum drag for any drag law written within the framework of the model of local interaction for a specified base area and maximum permissible length of the body was solved in [1] without simplifying assumptions regarding the body geometry. It was shown that it has an infinite set of solutions. The bodies constructed were called absolutely optimum bodies, since they all have the same drag, which, for a specified base area, cannot be less. The shape of the absolutely optimum body is formed by combinations of parts of the surface of a circular cone and planes, the normal to which makes a certain optimum angle with the direction of motion. This angle is determined by the characteristics of the medium and the body velocity in terms of constants which occur in the drag law. Absolutely optimum bodies were constructed previously [1] with a conical longitudinal contour and it was shown that they can have both a symmetrical and an asymmetrical cross-section.

Below, non-conical three-dimensional absolutely optimum bodies are constructed for a specified base area and maximum permissible length of the body, and the force characteristics of asymmetrical absolutely optimum bodies are investigated.

1. PROPERTIES OF THE OPTIMUM SURFACES

Consider the motion of a body in a medium with constant velocity directed opposite to the direction of the Ox axis, in a Cartesian rectangular system of coordinates $Oxyz$.

The force acting on the body can be written in the form

$$\mathbf{F} = q\iint_S (c_p\mathbf{n} + c_\tau\boldsymbol{\tau})dS \tag{1.1}$$

Here q is the velocity head, c_p and c_τ are the pressure coefficient and the coefficient of friction on the body surface, $\mathbf{n}$ and $\boldsymbol{\tau}$ are unit vectors of the inward normal and the tangent to an element of the surface, and the vectors $\mathbf{n}$, $\boldsymbol{\tau}$ and $\mathbf{x}$ are coplanar

$$\boldsymbol{\tau} = [[\mathbf{n}\times\mathbf{x}]\times\mathbf{n}]/|[\mathbf{n}\times\mathbf{x}]| \tag{1.2}$$

The integration in (1.1) is carried out over the surface S of contact between the medium and the body, for which

$$\alpha = (\mathbf{nx}) \geqslant 0, \quad (\boldsymbol{\tau}\mathbf{x}) = \sqrt{1-\alpha^2} \tag{1.3}$$

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Within the framework of the model of local interaction, each element of the body surface interacts with the medium independently of other parts of the surface, and the coefficients, c_p and c_τ are functions of α

$$c_p = c_p(\alpha), \quad c_\tau = c_\tau(\alpha) \tag{1.4}$$

In the general case, the body velocity and the characteristics of the medium, which are assumed constant, can occur in expression (1.4).

Suppose the body possesses a piecewise-smooth surface S and condition (1.3) is satisfied at each point of the surface, apart from a finite number of lines of derivative discontinuity. The base area of the body S_b is assumed specified. It is related to S as follows:

$$S_b = \iint_S \alpha dS \tag{1.5}$$

We will write the drag of the body in the form

$$F_1 = (\mathbf{Fx}) = q\iint_S f(\alpha)\alpha dS \tag{1.6}$$

$$f(\alpha) = c_p(\alpha) + (c_\tau(\alpha)/\alpha)\sqrt{1-\alpha^2}$$

Note that the body surface projects uniquely onto the plane of the base with the exception of parts with $\alpha = 0$. Denoting the overall area of these parts by S_0, we can represent integral (1.6) in the form

$$F_1 = q\left(\iint_{S_b} f(\alpha)dS_b + c_\tau(0)S_0\right)$$

We consider $f(\alpha)$ as a function of the real variable α in the section [0, 1]. Suppose it reaches a minimum for a certain value of $\alpha = \alpha^*$. Then, the following limit holds for the integral F_1

$$F_1 \geqslant F_1^* = qS_b f(\alpha^*)$$

The equality here can only occur when $\alpha \equiv \alpha^*$, when drag functional (1.6) reaches its absolute minimum F_1^*.

Hence, for a specified base area S_b one cannot obtain a drag less than F_1^*. The bodies which have such a drag are those at each point of the surface of which the following condition is satisfied

$$\alpha = \alpha^* = \text{const} \tag{1.7}$$

As in [1] we will call such bodies absolutely optimum bodies. It was shown in [1] that if the body surface in a cylindrical system of coordinates (ρ, x, θ) with origin of coordinates x at the vertex of the body is given by the equation

$$\rho = \psi(x, \theta) \tag{1.8}$$

the function $\psi(x, \theta)$, which gives the shape of the absolutely optimum body, satisfies the following equation on the smooth parts of the surface

$$\psi_x / [1 + (\psi_\theta / \psi)^2]^{1/2} = t^*, \quad t^* = \alpha^* / \sqrt{1-\alpha^{*2}} \tag{1.9}$$

When $\alpha^* < 1$ its solutions with $\psi_x = t^*$ and $\psi_\theta = 0$ define surfaces of circular cones, in particular, a cone with the function $\psi(x, \theta) = t^* x$ with origin of coordinates at its vertex. All the planes, tangential to this, also satisfy Eq. (1.9). The surface of a body consisting of combinations of parts of this cone and planes tangential to it, is given by the function

$$\psi(x, \theta) = \varphi(x)R(\theta) \tag{1.10}$$

where the continuous functions $\varphi(x)$ and $R(\theta)$ respectively define the longitudinal and transverse contours of the body. As follows from (1.9), in this case the longitudinal contour of the optimum surface is conical

$$\varphi'(x) = t_k = \text{const} > 0 \tag{1.11}$$

and the function $R(\theta)$ satisfies the equation

$$R^2 /(R^2 + R'^2)^{1/2} = r_k = t^* / t_k = \text{const} \tag{1.12}$$

Its solutions are two types of arcs: the arc of a circle with $R(\theta) \equiv r_k$ and sections of a straight line tangential to a circle of radius r_k, with $R(\theta) \equiv r_k/\cos(\theta - \theta_1)$, where θ_1 is an integration constant. The transverse contour of the optimum body may contain sections of different straight lines with different θ_1, continuously joining one another. This leads to the occurrence of lines of derivative discontinuity on the surface and may be the reason why the transverse contour of the optimum body will be asymmetrical. Condition (1.5) gives the rule for the joining of the sections of the straight lines, and for a body of length x_k with surface (1.8) can be written in the form

$$\int_0^{2\pi} \psi^2(x_k, \theta)d\theta = 2S_b \tag{1.13}$$

According to this, the following conditions must be satisfied for the functions $\varphi(x)$ and $R(\theta)$

$$\varphi(0) = 0, \quad \varphi(x_k) = (S_b / \pi)^{1/2} \tag{1.14}$$

$$R(0) = R(2\pi), \quad \int_0^{2\pi} R^2(\theta)d\theta = 2\pi \tag{1.15}$$

The surface of the absolutely optimum body which satisfies relations (1.1), (1.15) is conical. As follows from conditions (1.12) and (1.15), the value $r_k \leqslant 1$ and always $t_k \geqslant t^*$. This, in particular, indicates that a circular cone with $x_k^* = (S_b/\pi)^{1/2}/t^*$ has the greatest length x_k^* in the class of conical absolutely optimum bodies.

Examples of the transverse contours of conical absolutely optimum bodies projected onto the Oyz plane are shown in Fig. 1. Here contour 1 does not contain the arc of a circle, whereas arc AB of circle 3 of radius r_k belongs to contour 2. The straight lines, the sections of which are used to construct contours 1 and 2, are tangential to it. For the same base areas S_b and longitudinal contour $\varphi(x) = t_k x$ and $t_k = t^*/r_k$, bodies with cross-sections 1 and 2 have the same drag F_1^*. Note that they are asymmetrical about the horizontal plane Oxz.

It was shown earlier [1] that bodies of different length and different transverse dimensions, which may change continuously, without changing the drag of the body, belong to a class of conical absolutely optimum bodies for a specified base area S_b. We will show that in the class of absolutely optimum bodies one can construct an infinite set of non-conical bodies, the surface of which consists of parts of the surface of conical bodies which satisfy condition (1.7) and are continuously joined to one another.

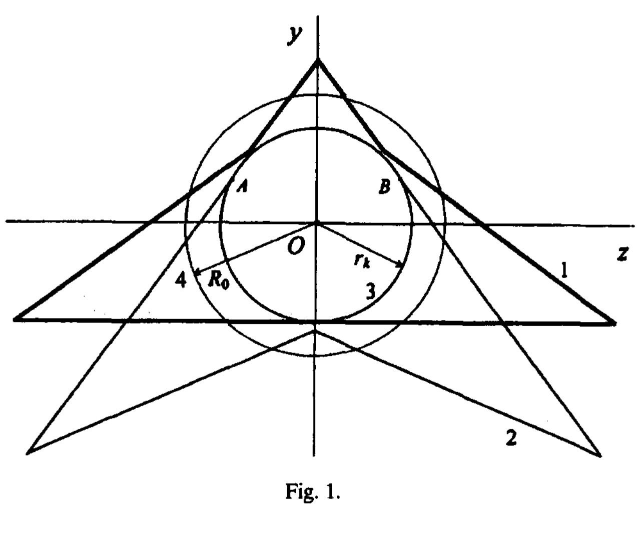

Fig. 1.

2. THE CONSTRUCTION OF NON-CONICAL ABSOLUTELY OPTIMUM BODIES

We will write the relation for $R(\theta)$ of a cyclically symmetrical ("star-shaped") conical absolutely optimum body, the transverse contour of which consists of N-similar symmetrical cycles. For known $r_k \leqslant 1$ and any $N \geqslant 2$ the transverse contour of such a body is completely defined by the function $R(\theta)$ in a half-cycle in the section $[0, \pi/N]$. It can consist of two arcs, smoothly joined at the point $\theta = \theta_0$

$$\begin{aligned} R(\theta) &\equiv r_k, \quad 0 \leqslant \theta \leqslant \theta_0 \\ R(\theta) &= r_k/\cos(\theta - \theta_1), \quad \theta_0 \leqslant \theta \leqslant \pi/N \end{aligned} \tag{2.1}$$

where the values of θ_0 and θ_1, taking conditions (1.15) into account, can be found from the relations

$$\operatorname{tg}(\theta_0 - \theta_1) = \operatorname{ctg}\left(\frac{\pi}{N} - \theta_0\right) - \frac{Nr_0^2}{(\pi - N\theta_0 r_0^2)}, \quad r_0 = \frac{r_k}{\cos(\theta_0 - \theta_1)} \tag{2.2}$$

Here r_0 is the minimum value of $R(\theta)$, where, without loss of generality, were assume that $r_0 = R(0)$. If

$$r_k \leqslant [(\pi/N)/tg(\pi/N)]^{1/2} \tag{2.3}$$

there is no arc of a circle in the transverse contour ($\theta_0 = 0$). If condition (2.3) is violated $r_k < 1$, the contour contains both arcs (2.1). In this case $\theta_1 = \theta_0$ and $r_0 = r_k$. For known r_k and N the values of r_0, θ_0 and θ_1 are defined uniquely from relations (2.2).

It was proved in [1] that if, when formulating the problem of the body of minimum drag, additional limitations are imposed on the length and number of cycles of the transverse contour of the required body, we can always choose a conical star-shaped absolutely optimum body which satisfies these and which will be the solution of the problem with these constraints. Using this property, for specified S_b, N and the maximum permissible length of the body L, we will construct a non-conical absolutely optimum body, whose surface consists of parts of a surface of two conical absolutely optimum bodies, joined continuously to one another.

We will take as one of the bodies a circular cone, which, in the class of absolutely optimum bodies, has the greatest length x_k^*. The surface of this cone is given by the function

$$\psi_1(x, \theta) = t^* x \tag{2.4}$$

Consider its cross-section for a certain $x = x_1 < \min(L, x_r^*)$. We will denote the radius of this cross-section, referred to $(S_b/\pi)^{1/2}$, by $R_0 = x_1/x_r^*$. For specified S_b and N we will also construct a conical star-shaped absolutely optimum body of length $x_2 < x_1$. Its longitudinal contour, by expressions (1.11) and (1.14), is defined by the function $\varphi(x) = t_k x$, where $t_k = (S_b/\pi)^{1/2}/x_2$, while the transverse contour consists of arcs of a circle of radius $r_k = x_2/x_r^*$ and sections of straight lines, that are tangent to it. The function of the transverse contour $R(\theta)$ is given by relations (2.1). We will shift this body along the Ox axis, transferring its base to the point $x = x_1$ (see Fig. 2a). After this shift, the body surface is given by the function

$$\psi_2(x, \theta) = t_k(x - x_0)R(\theta) \tag{2.5}$$

where $x_0 = x_1 - x_2$, which obviously satisfies Eq. (1.9) and condition (1.13) when $x_k = x_1$. The surface of this absolutely optimum body intersects the surface of circular cone (2.4) along certain three-dimensional curves.

Consider a new body of length x_1 whose surface consists of combinations of parts of the surfaces of circular cone (2.4) and star-shaped absolutely optimum body (2.5). We will define its function $\psi(x, \theta)$ for any $\theta \in [0, 2\pi]$ as follows:

$$\begin{aligned} \psi(x, \theta) &= \psi_1(x, \theta), \quad 0 \leqslant x \leqslant x_0 \\ \psi(x, \theta) &= \max(\psi_1(x, \theta), \ \psi_2(x, \theta)), \quad x_0 \leqslant x \leqslant x_1 \end{aligned} \tag{2.6}$$

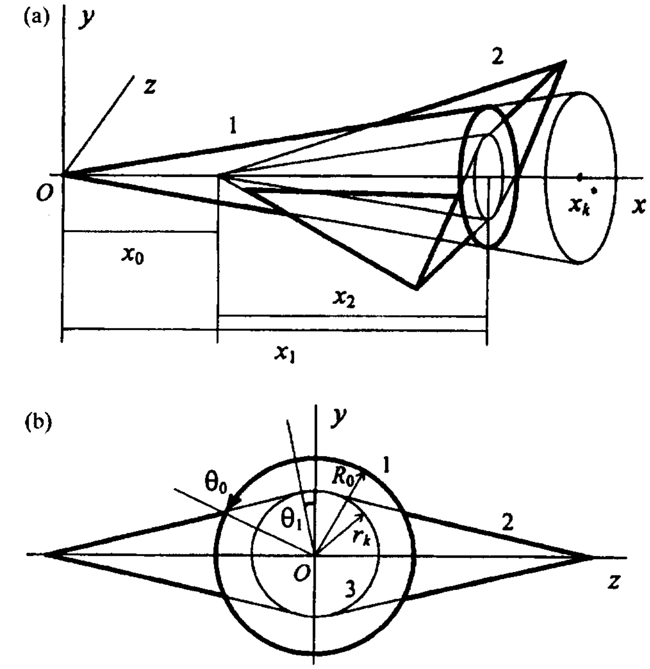

Fig. 2

This function, like $\psi_1(x, \theta)$ and $\psi_2(x, \theta)$, is piecewise-smooth and over the whole region of definition, apart from a finite number of lines of derivative discontinuity, satisfies Eq. (1.9). It specifies the surface of the absolutely optimum body if, when $x_k = x_1$, condition (1.13) is satisfied for it.

Thus, if $R_0 \leqslant r_0$, where r_0 is found from relations (2.2) for known r_k and N, the required non-conical absolutely optimum body has been constructed. In this case, in the plane of the base for all $\theta \in [0, 2\pi]$ the function $\psi(x_1, \theta) = \psi_2(x_1, \theta)$ and condition (1.13) is satisfied automatically. If $R_0 > r_0$, which is always true when condition (2.3) is violated, in particular, this will always be the case when $N = 2$, the body constructed has the base area which is greater than that specified. In this case, the body base has the form shown in Fig. 2(b). Its contour consists of combinations of arcs of circle 1 and sections of straight lines 2, which belong to surfaces 1 and 2 respectively of the conical absolutely optimum bodies shown in Fig. 2(a). The base area of this body, without violating optimality condition (1.7), can be reduced by increasing the angle θ_1, which is the integration constant of Eq. (1.12). For a fixed longitudinal contour (1.11) and retaining the value of r_k, the solution of Eq. (1.12) with a new integration constant defines the function (2.5) which, as previously, will satisfy Eq. (1.9). The new value of θ_1 can be found from relations (2.2) if we replace r_0 by R_0 in it. Then, θ_0 and θ_1 are determined uniquely from them and also, of course, the contour of the body base. In a half cycle in the section $[0, \pi/N]$ it will consist of two arcs which touch at the point $\theta = \theta_0$

$$R(\theta) \equiv R_0, \quad 0 \leqslant \theta \leqslant \theta_0$$
$$R(\theta) = r_k/\cos(\theta - \theta_1), \quad \theta_0 \leqslant \theta \leqslant \pi/N$$

Since θ_0 and θ_1 were chosen from condition (1.13), the non-conical body constructed will belong to the class of absolutely optimum bodies, since the function (2.6), which specifies its surface, now satisfies conditions (1.9) and (1.13). As follows from the rule for constructing it, one can take any number of cycles N of the star-shaped absolutely optimum body, while the values of x_1 and x_2 need only satisfy the condition: $x_2 < x_1 < \min(L, x_k^*)$. By changing them one can obtain non-conical absolutely optimum bodies, which, like conical bodies, will have a different length and different transverse dimensions. For an equal area of the base S_b they will all have the same drag F_1^*.

In Fig. 2(a), as an example we show the shape of a non-conical absolutely optimum body for $N = 2$. In practice, this shape may turn out to be more preferably conical. Thus, for example, in aerodynamics it is called [2] an aerodynamically perfect body. In particular, one can assume that the motion of non-conical absolutely optimum bodies in the medium will be more stable than the motion of conical bodies, since the centre of the pressure force acting on the body for small angles of attack is displaced backwards compared with conical absolutely optimum bodies.

By combining parts of the surface of circular cone (2.4) and parts of the planes tangent to it, one can construct an absolutely optimum body which satisfies the most diverse practical requirements. These may be constraints on the transverse dimensions of the body and the requirement that its surface should pass through a specified contour of the base.

Consider, for example, the problem of constructing a body of minimum drag, when, apart from S_b and L, one is given the shape of its base in the form of a circle. We will show that there is always a non-conical absolutely optimum body whose surface satisfies specified conditions.

The surface of the required body will consist of parts of the surfaces of two conical bodies, one of which is the circular (2.4) of length x_k^*. The second, a star-shaped body of length $x_k \leqslant \min(L, x_k^*)$ and consisting of N symmetrical cycles will be constructed in accordance with the following rule. We consider a cross-section of cone (2.4) at $x = x_k$. We will denote its radius, referred to $(S_b/\pi)^{1/2}$, by $r_k = x_k/x_k^* \leqslant 1$. Depending on the value of r_k we take the solution of Eq. (1.12) $R(\theta) = r_k/\cos(\theta - \theta_1)$ with integration constant $\theta_1 = -\arccos(r_k/r_0)$, where $r_0 = R(0)$ can take any value from the condition $r_0 \geqslant 1$. Using this solution for the function $R(\theta)$ in the half-cycle in the section $[0, \pi/N]$, we construct the star-shaped transverse contour of the body. It is possible to construct the contour if $N > \pi/\arcsin(r_k/r_0)$. We take it as the base contour of a conical body of length x_k, the longitudinal contour of which $\varphi(x) = t_k x$, where $t_k = t^*/r_k$. The function $\psi_2(x, \theta) = \varphi(x)R(\theta)$ specifies its surface and satisfies Eq. (1.9) on the smooth parts. The body surface consists of parts of planes which touch the cone (2.4), and condition (1.7) is satisfied on these. The minimum radius of its base is $r_0(S_b/\pi)^{1/2}$.

We shift the body along the Ox axis, transferring its base to the point $x = x_k^*$ (see Fig. 3). The vertex of the body is shifted to the point $x = x_0 = (x_k^* - x_k)$, while the circle of minimum radius of the base at $r_0 = 1$ coincides with the contour of the base of cone (2.4). Here, as shown in Fig. 3, the surfaces of cone 1 and of the star-shaped body 2 intersect along certain three-dimensional curves.

Consider a new body of length x_k, whose surface when $x \in [x_0, x_k^*]$ for each $\theta \in [0, 2\pi]$ is defined by the function $\psi(x, 0) = \min(\psi_1(x, \theta), \psi_2(x - x_0, \theta))$. The optimality condition (1.7) is satisfied on the smooth parts of its surface, while the contour of the base is a circle of radius $(S_b/\pi)^{1/2}$. The surface of this body satisfies all the specified conditions and, consequently, it is the required absolutely optimum body. An example of its shape for $r_0 = 1$ is shown in Fig. 3 in the form of configuration 3. In the region of the vertex the surface of the body is given by the function $\psi_2(x - x_0, \theta)$. This denotes that the spout of the body is star-shaped and on a section at the vertex of length $\Delta x_1 = x_k(1 - r_k)/(r_f - r_k)$, where $r_f = r_k/\cos(\pi/N - \theta_1)$, its transverse contour consists solely of sections of straight lines. On moving away from the vertex, the transverse contour arcs of circles appear, belonging to cone 1. When $r_0 > 1$

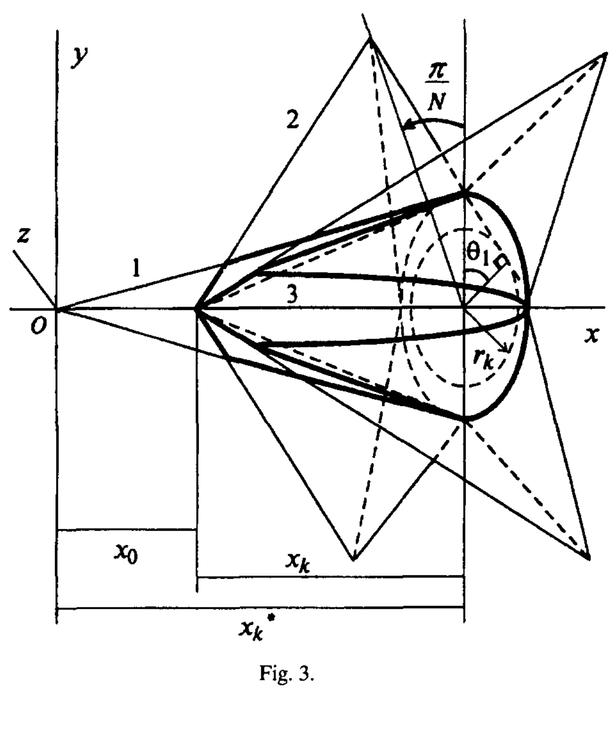

Fig. 3.

near the base on a section of length $\Delta x_2 = x_k(r_0 - 1)/(r_0 - r_k)$ the body will have an axisymmetrical part, belonging completely to the surface of cone (2.4). Taking into account the fact that, for specified S_b and L, there is arbitrariness in choosing the values of x_k, r_0 and N, then, by varying these, we obtain an infinite set of absolutely optimum bodies, the contour of the base of which is a circle.

In practice, an absolutely optimum body with a circular base may be the head part of a specified length of a certain body of revolution. As is well known, when a body moves in dense media, such as soil and metal, the stresses acting on its surface often lead to the body deformation and fracture. The strength of an absolutely optimum body with a circular base is higher than that of conical absolutely optimum bodies of equivalent length and base area. Hence, for these media, when choosing the optimum configuration, such absolutely optimum bodies will be more preferable than conical.

We have given above examples of non-conical absolutely optimum bodies constructed from two conical bodies, on the surface of which the optimality condition (1.7) is satisfied. In a similar way one can construct absolutely optimum bodies consisting of parts of surfaces of several bodies satisfying (1.7).

The examples considered demonstrate the possibility of choosing the optimum configuration of the class of absolutely optimum bodies when quite different conditions are specified regarding the body geometry. These also include asymmetrical bodies. They can be conical, and their surface is then described by relations (1.10)–(1.15). The surfaces of the non-conical bodies can be made up of parts of the surfaces of asymmetrical absolutely optimum bodies, like parts of symmetrical bodies, in accordance with the rules described above. For example, the shape of the absolutely optimum body shown in Fig. 2(a) can be made asymmetrical if when constructing it one uses a conical shape with transverse contour 1 or 2, shown in Fig. 1, as the second surface. Then, according to rule (2.6) for constructing a new surface in the plane of the base of the body at $x = x_1$ the transverse contour of the absolutely optimum body, in addition to the parts of contours 1 or 2, will contain arcs of circle 4 (Fig. 1). Asymmetrical absolutely optimum bodies, like symmetrical bodies, for equal base areas S_b will all have the same drag F_1^*. We will investigate their force characteristics in a plane perpendicular to the direction of motion.

3. THE FORCE CHARACTERISTICS OF ASYMMETRICAL ABSOLUTELY OPTIMUM BODIES

We will show that within the framework of the local-interaction model, when a body moves along the Ox axis, on the surface of which conditions (1.3) and (1.7) are satisfied, irrespective of the shape of the cross-section and the form of functions (1.4), the force acting on the body has no component in the Oyz plane.

We will assume that this is not so and that the force (1.1) in the Oyz plane has a non-zero component directed along a certain unit vector $\mathbf{e}$. Then

$$\mathbf{F} = F_1\mathbf{x} + F_2\mathbf{e} \tag{3.1}$$

Using expressions (1.2) and (1.3), we will write expansions of the vectors $\mathbf{n}$ and $\boldsymbol{\tau}$ in the $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ basis in the form

$$\mathbf{n} = \alpha\mathbf{x} + n_y\mathbf{y} + n_z\mathbf{z}, \quad \boldsymbol{\tau} = \beta\mathbf{x} - t(n_y\mathbf{y} + n_z\mathbf{z})$$
$$\beta = \sqrt{1-\alpha^2}, \quad t = \alpha/\beta$$

Here and henceforth the asterisk on α is omitted.

Since $(\mathbf{xe}) = 0$, it follows that

$$(\boldsymbol{\tau}\mathbf{e}) = -t(\mathbf{ne}) \tag{3.2}$$

Consider the component F_2 in representation (3.1). Taking relations (1.1), (1.7) and (3.2) into account we can write the expression for this component in the form

$$F_2 = (\mathbf{Fe}) = q(c_p(\alpha) - tc_\tau(\alpha))\iint\limits_S (\mathbf{ne})\,dS$$

Since $(\mathbf{ne}) = 0$ on the perpendicular axis Ox of the body base, in the latter expression the integral over the "windward" part of the body S can be replaced by an integral over its whole surface. Applying Gauss' formula to this and taking into account the constancy of the vector $\mathbf{e}$, we obtain that this integral, and together with it the component of the force F_2, are equal to zero.

Hence, if the body possesses a piecewise-smooth surface, on which conditions (1.3) are satisfied, while α is a constant, within the framework of the local-interaction model, irrespective of the form of functions (1.4), the force acting on the body when it moves along the Ox axis, has no component in a plane perpendicular to the direction of motion. The result obtained is independent of the specific value of α. This denotes, in particular, that bodies with any conical longitudinal contour, the transverse sections 1 and 2 of which are shown in Fig. 1, when moving along the Ox axis experience no lifting and lateral forces.

4. CONCLUSION

For a specified length and base area of the body, using optimality condition (1.7) we have developed a method of constructing three-dimensional shapes of minimum drag, which enables optimum bodies of various configuration to be constructed, including non-conical an asymmetrical bodies. It is based on a continuous joining of parts of the surface of several conical bodies, which satisfy condition (1.7). A method of constructing conical bodies when different conditions are imposed on their geometry is known [1], and their surface consists of parts of the surface of a circular cone with aperture angle $\beta = 2 \arcsin(\alpha^*)$ and planes touching it. As a result, the surface of the new body consists of these parts, but it can be non-conical and asymmetrical. All the bodies constructed have the same drag F_1^*, and a smaller value of this cannot be obtained for a specified base area. As previously [1], such bodies are called absolutely optimum bodies. We have proved that the force acting on the optimum body, even when it has no symmetry, has no component in a plane perpendicular to the direction of motion. We have considered examples of the construction of bodies of minimum drag, which demonstrate the possibilities of the procedure for obtaining optimum shapes, when additional constraints are imposed on the body geometry. We have shown that it can be used to construct absolutely optimum bodies of specified length when the shape of the base is given in the form of a circle.

At first glance, solutions extremely close to those obtained above were obtained earlier in [3, 4]. Assuming that the pressure is given by Newton's formula and the friction is constant [3] or nonexistent [4], sought a body surface of minimum drag $x = x(y, z)$, passing through a specified base contour was. When there is no friction [4] (see also [5]) it was assumed that the area of the body surface S is bounded by a certain constant surface S^*. For an arbitrary body length, a condition for the optimum surface was obtained in [3, 5] similar to the optimality condition (1.7). It was given in the form of a partial differential equation

$$x_y^2 + x_z^2 = \text{const} \tag{4.1}$$

Since $\alpha = 1/[1 + x_y^2 + x_z^2]^{1/2}$, Eq. (4.1) is equivalent to the condition that α in (1.7) should be constant (4.1). when calculating the constant in (4.1) the following expressions were obtained (in the notation used in the present paper): $(1/t^*)^2$ [3] and $((S^*/S_b)^2 - 1)$ [4, 5]. Despite this, the results obtained above, and also in [1], on the one hand, and the results obtained in [3–5] on the other, differ in principle.

The require integral surface [4, 5] of Eq. (4.1) was constructed as a surface of constant slope, and the optimum surface in [3] is given in parametric form as the solution of Eq. (4.1), obtained by the standard method of characteristic strips [6]. When integrating Eq. (4.1) to obtain the closed surface of the body passing through a specified base contour, this method can be used for a piecewise-smooth contour, but only if the internal angle at which the arcs meet at points of derivative discontinuity are less than 180°. Otherwise, the characteristic strips diverge and no closed surface is obtained. It can be closed by using the surface of the characteristic conoid with vertex at the derivative discontinuity point, which, for Eq. (4.1), is a circular cone with $\beta = 2 \arcsin(\alpha^*)$, but the length of the body in this case may be much greater than the permissible value.

Comparing the solutions obtained above with the solution proposed in [3], we note that, using the latter, one can construct conical star-shaped absolutely optimum bodies if the contour of their base is convex. However, if the contour of the base of the absolutely optimum body is concave, as, for example, for bodies the transverse contours of which are shown in Figs 1 and 2, or like the star-shaped absolutely optimum bodies [1] when condition (2.3) is satisfied in the form of a strict inequality, one cannot construct a conical absolutely optimum body by the method which was used in [3].

The difference in the solutions can be most clearly seen when constructing the optimum body having a base in the form of a circle. In this case the method of characteristic strips enables Eq. (4.1) to be integrated and a unique solution can be obtained [7]. This is the surface of a circular cone with an angle at the vertex of $\beta = 2 \arcsin(\alpha^*)$. The construction of the optimum surface, like the surfaces of constant slope [4, 5], gives the same result in this case with $\alpha^* = S_b/S^*$. However, as was shown above, in the class of piecewise-smooth functions $x = x(y, z)$ an infinite set of solutions exists which define the body surface, on the smooth parts of which condition (4.1) is satisfied. An example of such a surface is given in Fig. 3 as the shape 3.

Hence, we have shown that the majority of solutions of the problem of the shape of the body of minimum drag, obtained above and previously [1], cannot be constructed by the method of characteristic strips [6, 7], which was used in [3], or as a constant-slope surface [4, 5]. On can obtain them thanks to the singular properties of Eq. (4.1), for which a circular cone with an angle at the vertex $\beta = 2$ arcsin (α^*) and an axis directed along Ox is a characteristic conoid. All the planes which touch it are integral surfaces of Eq. (4.1), and this enables one to use parts of them to construct optimum bodies. This property of a cone and planes was the basis of the method of obtaining absolutely optimum bodies above.

The results of theoretical and experimental research on reducing the drag of multiwedge and pyramidal bodies, among which there are shapes whose structure is close to absolutely optimum bodies, and a review of which is given in [8] for various media, confirm the advisability of changing to such shapes for velocities of motion of the body in a medium of the order of 10^2–10^3 m/s. However, it should be noted that the conclusion in [8] that numerical methods have the prerogative in solving the problem of the three-dimensional shape of a body of minimum drag within the framework of local theories and that it can only be obtained by narrowing the class of permissible surfaces is refuted by the results of this paper. For specified lengths and base areas of the body the procedure for obtaining the absolutely optimum bodies enables three-dimensional configurations of minimum drag to be constructed in the class of piecewise-smooth surfaces without using numerical methods.

In conclusion we note that the optimization of the shape of the body and the calculation of the force characteristics were presented above within the framework of the local interaction model, which ignores secondary collisions between the particles of the medium and the body and their interaction with one another after reflection from the body surface. However, these collisions and interactions will always occur if the body surface has lines of derivative discontinuity. In this case, an additional investigation of the force characteristics of the bodies is required based on experiment and a more rigorous theory [2].

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